Phase transitions and hysteresis: new perspectives and results

Michela Eleuteri

Università degli Studi di Milano

Supported by the FP7-IDEAS-ERC-StG Grant “EntroPhase” #256872 (P.I. E. Rocca)

Giornata di Studio “Prospettive di sviluppo della matematica applicata in Italia 2013” - Workshop SIMAI Giovani 2013

Roma - March 11, 2013
Plan of the talk

- **Hysteresis: a rate-independent memory effect**
  - The stop and the Prandtl-Ishlinskii operators
  - New theory of oscillating elastoplastic beams and plates
- **Motivation for material fatigue**
- **Evolution equation for the fatigue**
- **The model with phase transition**
- **Thermodynamical consistency**
- **Conclusion**
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Hysteresis: a rate-independent memory effect

- **Hysteresis**: a rate-independent memory effect (multidisciplinary character)

Tipical hysteresis diagram in ferromagnetism ($h$ magnetic field, $m$ magnetization).

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The stop and the Prandtl-Ishlinskii operators

\[ \sigma = \varphi_2 \delta_{r_2}[\epsilon] \]

\[ \sigma = \varphi_1 \delta_{r_1}[\epsilon] \]

\[ \sigma = \varphi_1 \delta_{r_1}[\epsilon] + \varphi_2 \delta_{r_2}[\epsilon] \]
A classical hysteresis-type model for 1D elastoplasticity

- Introduced by L. Prandtl and A. Yu. Ishlinskii (extensions to the multidimensional case are possible)
- The relation between (one-dimensional) strain $\varepsilon$ and stress $\sigma$ is given in the form of the so-called Prandtl-Ishlinskii operator

$$\sigma = \mathcal{P}[\varepsilon](t) = \int_0^\infty s_r[\varepsilon](t) \varphi(r) \, dr$$

for all $\varepsilon \in W^{1,1}(0, T)$. Here $\varphi > 0$ is a nonnegative weight function not known a priori and $s_r$ represents the one-dimensional elastic-ideally plastic element or stop operator, with the threshold $r > 0$

- Prandtl-Ishlinskii description of elastoplasticity: a superposition of infinitely many stop operators having different thresholds (very imaginative and easily understood) BUT engineers very often prefer classical engineering approaches like the three-dimensional von Mises or Tresca models

- Motivation: the disadvantage that the weight function $\varphi$ is not known a priori and must be identified
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New theory of oscillating elastoplastic beams and plates


**Key point**: the 3D single-yield von Mises criterion leads after a dimensional reduction to a multi-yield Prandtl-Ishlinskii operator where the weight function $\varphi$ can be explicitly determined!

A plate section with grey plasticized zone.
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Motivation for material fatigue

- Plastic deformations lead to energy dissipation and material fatigue, manifested by material softening, heat release, material failure in finite time.

- Very important: take into account the effects of energy exchange and estimating the lifetime of oscillating thermoelastoplastic structures under material fatigue.

- **Aim:** develop a thermodynamically consistent theory of oscillating thermoelastoplastic plates under material fatigue (dynamic approach - different from literature).

- The resulting system from the theory developed by Krejčí & al:

\[
\begin{align*}
\partial_{tt} w - \partial_{tt} \Delta w + D^*_2 \sigma &= g, \\
\sigma &= B \varepsilon + \int_0^\infty s_r Z[\varepsilon](t) \varphi(r) \, dr, \\
\varepsilon &= D_2 w
\end{align*}
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- We introduce \( \theta > 0 \) (absolute temperature) and \( m(x,t) \geq 0 \) (material fatigue).
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\[\partial_{tt} w - \partial_{tt} \Delta w + \mathbf{D}_2^* \sigma = g,\]

\[\sigma = \mathbf{B}(m) \varepsilon + \int_0^\infty s_r Z[\varepsilon](t) \varphi(\theta, r) dr - \beta (\theta - \theta_c) \mathbf{1}\]

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- We introduce $\theta > 0$ (absolute temperature) and $m(x,t) \geq 0$ (material fatigue); **aim:** get an evolution equation for $m$ consistent from the thermodynamic point of view
Evolution equation for the fatigue

- **Main assumption:** proportionality between rate of fatigue $\partial_t m$ and

$$\mathcal{D} = \langle \sigma, \partial_t \varepsilon \rangle - \partial_t \theta \mathcal{I}[\theta, \varepsilon] - \partial_t \mathcal{F}[\theta, \varepsilon]$$

$$= -\frac{1}{2} \langle B'(m) \varepsilon, \varepsilon \rangle \partial_t m + \int_{0}^{\infty} \langle \partial_t (\varepsilon - s_r Z[\varepsilon]), s_r Z[\varepsilon] \rangle \varphi(\theta, r) \, dr$$

where $\mathcal{F}$ is the specific free energy and $\mathcal{I}$ is the specific entropy

- Justified by the so-called **rainflow method for cyclic fatigue accumulation** in uniaxial processes (counts closed hysteresis loops in the loading history - mechanism of energy dissipation)

- In multiaxial loading processes? Experimental measurements at the point of material failure: strong temperature increase, manifested by energy dissipation peak (temperature tests are in engineering practice for damage analysis in high frequency regimes (e.g. in aircraft industry))

$$\left( \frac{1}{C(\theta)} + \frac{1}{2} \langle B'(m) \varepsilon, \varepsilon \rangle \right) \partial_t m = \int_{0}^{\infty} \langle \partial_t (\varepsilon - s_r Z[\varepsilon]), s_r Z[\varepsilon] \rangle \varphi(\theta, r) \, dr$$

- $B'(m) \leq 0$ softening $\Rightarrow$ singularity! Material failure in finite time!
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The model with phase transition

**Motivation:**
- possibility to account also for decreasing fatigue rate (in view of engineering applications)
- the material can be partially repaired by local melting

**How to achieve this goal:**

**Phase transition equation in the form of melting-solidification law**

\[
\gamma \chi_t \in - \partial \chi \mathcal{F}[\varepsilon, \theta, \chi] \quad \chi \in [0, 1]
\]

\[
\chi_0 \in [0, 1] \text{ some initial condition, } A(x, t) := \int_0^t \frac{1}{\gamma} \left( \frac{L}{\beta_c} (\theta - \theta_c) \right) (x, \tau) d\tau
\]

\[
(\chi_t - A_t)(z - \chi) \geq 0 \text{ for all } z \in [0, 1]
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- material fatigue and $\chi$: degree of melting
- the time of failure of the material can be shifted possibly considering a sufficiently large time interval of observation (usual engineering viewpoint) a global solution of the corresponding PDEs system can be found

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- Phase transition equation in the form of melting-solidification law

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\gamma \chi_t \in -\partial_\chi \mathcal{F}[\varepsilon, \theta, \chi] \quad \chi \in [0,1]
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\chi_0 \in [0,1] \text{ some initial condition, } A(x,t) := \int_0^t \frac{1}{\gamma} \left( \frac{L}{\theta_c} (\theta - \theta_c) \right) (x, \tau) d\tau
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(\chi_t - A_t)(z - \chi) \geq 0 \text{ for all } z \in [0,1]
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  \[ \chi \in \mathcal{S}_{[0,1]}[\chi_0, A] \quad \mathcal{S}_{[0,1]} \text{ is a shifted stop} \]
Thermodynamical consistency

- If we introduce $\mathcal{F}[\varepsilon, \theta, \chi]$ specific free energy, $\mathcal{S}[\varepsilon, \theta, \chi]$ specific entropy and $\mathcal{U}[\varepsilon, \theta, \chi]$ internal energy we are able to show that the first and second principles of thermodynamics are satisfied

\[
\frac{\partial}{\partial t} \mathcal{U}[\varepsilon, \theta, \chi] + \text{div} q = \langle \sigma, \varepsilon_t \rangle \quad \text{(energy conservation)}
\]

\[
\frac{\partial}{\partial t} \mathcal{S}[\varepsilon, \theta, \chi] + \text{div} \frac{q}{\theta} \geq 0, \quad \text{(Clausius-Duhem inequality)}
\]

- Evolution equation for $m$:

\[
(C - \langle B'(m)\varepsilon, \varepsilon \rangle) m_t = -h(\chi_t) + \int_{0}^{\infty} \langle \partial_t (\varepsilon - s_r Z[\varepsilon])s_r Z[\varepsilon] \rangle \varphi(\theta, r) \, dr
\]

- allow the possibility of decreasing rate (i.e. $m_t < 0$) but only in the case if $\chi$ grows faster than the plastic dissipation rate (strong melting)
- external heat source (nonlinear boundary condition)
Thermodynamical consistency

- If we introduce $F[\varepsilon, \theta, \chi]$ specific free energy, $S[\varepsilon, \theta, \chi]$ specific entropy and $U[\varepsilon, \theta, \chi]$ internal energy we are able to show that the first and second principles of thermodynamics are satisfied

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Conclusion

- Rainflow method for fatigue evaluation in elastoplastic materials (uniaxial cyclic loading) allow to consider dissipated energy as a measure for fatigue
- The solution cannot be expected to exist globally: singularities (thermal shocks) occur in finite time
- Phase transition in the model to account also for decreasing fatigue rate
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