Plane Wave DG Methods: Exponential Convergence of the $hp$-version

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The Helmholtz equation

Simplest model of linear & time-harmonic waves:

$$-\Delta u - \omega^2 u = 0$$

in bdd. $\Omega \subset \mathbb{R}^N$, $N = 2, 3$, $\omega > 0$, (+ impedance/Robin b.c.)
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(+ impedance/Robin b.c.)

Why is it interesting?

1. Very general, related to any linear wave phenomena:
   - wave equation: \(\frac{\partial^2 U}{\partial t^2} - \Delta U = 0\)
   - time-harmonic regime: \(U(\mathbf{x}, t) = \Re\{u(\mathbf{x})e^{-i\omega t}\}\) → Helmholtz equation;

2. plenty of applications;

3. easy to write...
The Helmholtz equation

Simplest model of linear & time-harmonic waves:

\[-\Delta u - \omega^2 u = 0\]  \hspace{1cm} \text{in bdd. } \Omega \subset \mathbb{R}^N, \; N = 2, 3, \; \omega > 0, \hspace{0.5cm} (+ \text{ impedance/Robin b.c.})

Why is it interesting?

1. **Very general**, related to any linear wave phenomena:
   - wave equation: \( \frac{\partial^2 U}{\partial t^2} - \Delta U = 0 \)
   - time-harmonic regime: \( U(\mathbf{x}, t) = \Re\{u(\mathbf{x})e^{-i\omega t}\} \) \( \rightarrow \) Helmholtz equation;

2. **plenty of applications**;

3. **easy to write. . . but difficult** to solve numerically \( (\omega \gg 1) \):
   - oscillating solutions \( \rightarrow \) approximation issue,
   - numerical dispersion / pollution effect \( \rightarrow \) stability issue.
Difficulty #1: oscillations

Time-harmonic solutions are inherently oscillatory: a lot of DOFs needed for any polynomial discretisation!

Wavenumber $\omega = 2\pi/\lambda$ is the crucial parameter ($\lambda =$ wavelength).
Big issue in FEM solution for high wavenumbers: pollution effect

\[ \frac{\| \text{Galerkin error} \|}{\| \text{best approximation error} \|} \geq C \omega^a \quad a > 0, \quad \omega \to \infty. \]

It affects every (low order) method in \( h \): (BABUŠKA, SAUTER 2000).
Big issue in FEM solution for high wavenumbers: pollution effect

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\]

It affects every (low order) method in \( h \): (Babuška, Sauter 2000).

\[\downarrow\]

Oscillating solutions + pollution effect = standard FEM are too expensive at high frequencies!

Special schemes required, \( p \)- and \( hp \)-versions preferred.

Zienkiewicz, 2000: “Clearly, we can consider that this problem remains unsolved and a completely new method of approximation is needed to deal with the very short-wave solution.”
Trefftz methods

Piecewise polynomials used in FEM are “general purpose” functions, can we use discrete spaces tailored for Helmholtz?

Yes: Trefftz methods are finite element schemes such that test and trial functions are solutions of the Helmholtz equation in each element $K$ of the mesh $\mathcal{T}_h$, e.g.:

$$V_p \subset T(\mathcal{T}_h) = \left\{ v \in L^2(\Omega) : -\Delta v - \omega^2 v = 0 \text{ in each } K \in \mathcal{T}_h \right\}.$$

Main idea: more accuracy for less DOFs.
Typical Trefftz basis functions for Helmholtz

1. plane waves (PWs), \( \mathbf{x} \mapsto e^{i\omega \mathbf{x} \cdot \mathbf{d}} \) \( \mathbf{d} \in \mathbb{S}^{N-1} \)
2. circular / spherical waves (CWs),
3. corner waves,
4. fundamental solutions/multipoles,
5. wavebands,
6. evanescent waves, \ldots
Wave-based methods

Trefftz schemes require discontinuous functions. How to “match” traces across interelement boundaries?

Plenty of Trefftz schemes for Helmholtz, Maxwell and elasticity:

- **Least squares**: method of fundamental solutions (MFS), wave-based method (WBM);
- **Lagrange multipliers**: discontinuous enrichment (DEM);
- **Partition of unity method** (PUM/PUFEM), non-Trefftz;
- **Variational theory of complex rays** (VTCR);
- **Discontinuous Galerkin** (DG): Ultraweak variational formulation (UWVF).

We are interested in a family of Trefftz-discontinuous Galerkin (TDG) methods that includes the UWVF of Cessenat–Després.
• TDG method for Helmholtz: formulation and a priori ($p$-version) convergence
• Approximation theory for plane and spherical waves
• Exponential convergence of the $hp$-TDG
Part I

TDG method for the Helmholtz equation
Consider Helmholtz equation with impedance (Robin) b.c.:

\[-\Delta u - \omega^2 u = 0 \quad \text{in } \Omega \subset \mathbb{R}^N \text{ bdd., Lip., } N = 2, 3\]

\[\nabla u \cdot \mathbf{n} + i\omega u = g \quad \in L^2(\partial \Omega);\]
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2. introduce a mesh $\mathcal{T}_h$ on $\Omega$;
**TDG: derivation — I**

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2. Introduce a mesh $\mathcal{T}_h$ on $\Omega$;

3. Multiply the Helmholtz equation with a test function $v$ and integrate by parts on a single element $K \in \mathcal{T}_h$:

\[
\int_K (\nabla u \cdot \nabla \bar{v} - \omega^2 u \bar{v}) \, dV - \int_{\partial K} (n \cdot \nabla u) \bar{v} \, dS = 0;
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\]

4. integrate by parts again: ultraweak step

\[
\int_K (-u \Delta \overline{v} - \omega^2 u \overline{v}) \, dV + \int_{\partial K} (-n \cdot \nabla u \overline{v} + u n \cdot \nabla \overline{v}) \, dS = 0;
\]
choose a discrete Trefftz space $V_p(K)$ and replace traces on $\partial K$ with numerical fluxes $\hat{u}_p$ and $\hat{\sigma}_p$:

\[
\begin{align*}
  u &\rightarrow u_p \quad \text{(discrete solution)} \quad \text{in } K, \\
  u &\rightarrow \hat{u}_p, \quad \frac{\nabla u}{i\omega} \rightarrow \hat{\sigma}_p \quad \text{on } \partial K;
\end{align*}
\]
5. Choose a discrete Trefftz space $V_p(K)$ and replace traces on $\partial K$ with numerical fluxes $\hat{u}_p$ and $\hat{\sigma}_p$:

- $u \to u_p$ (discrete solution) in $K$,
- $u \to \hat{u}_p$, $\nabla u \to \hat{\sigma}_p$ on $\partial K$;

6. Use the Trefftz property: $\forall \nu_p \in V_p(K)$

$$\int_K u_p(-\Delta \nu_p - \omega^2 \nu_p) \, dV + \int_{\partial K} \hat{u}_p \nabla \nu_p \cdot n \, dS - \int_{\partial K} i\omega \hat{\sigma}_p \cdot n \, \nu_p \, dS = 0.$$  

TDG eq. on 1 element

Two things to set:
- discrete space $V_p$ and numerical fluxes $\hat{u}_p$, $\hat{\sigma}_p$. 
The abstract error analysis works for every discrete Trefftz space!

Possible choice: plane wave space

\[ V_p(T_h) = \left\{ v \in L^2(\Omega) : v|_K(x) = \sum_{\ell=1}^{p} \alpha_\ell e^{i\omega x \cdot d_\ell}, \alpha_\ell \in \mathbb{C}, \forall K \in T_h \right\}. \]

\( p \) := number of basis plane waves (DOFs) in each element.
Choose the numerical fluxes as:

\[
\begin{align*}
\hat{\sigma}_p &= \frac{1}{i\omega} \{ \nabla_h u_p \} - \alpha \lfloor u_p \rfloor_N & \text{on interior faces,} \\
\hat{u}_p &= \{ u_p \} - \beta \frac{1}{i\omega} \lfloor \nabla_h u_p \rfloor_N \\
\hat{\sigma}_p &= \frac{\nabla_h u_p}{i\omega} - (1 - \delta) \frac{1}{i\omega} (\nabla_h u_p + i\omega u_p n - g n) & \text{on } \partial \Omega. \\
\hat{u}_p &= u_p - \delta \frac{1}{i\omega} (\nabla_h u_p \cdot n + i\omega u_p - g)
\end{align*}
\]

\(\{ \cdot \} = \text{averages,} \quad \lfloor \cdot \rfloor_N = \text{normal jumps on the interfaces.}\)

\(\alpha, \beta > 0, \delta \in (0, \frac{1}{2}]\) parameters at our disposal (in \(L^\infty(\mathcal{F}_h)\)):

- \(h\)- or \(p\)-version, quasi-uniform meshes:
  - \(\alpha, \beta, \delta \) independent of \(\omega, h, p\); \(\text{UWVF: } \alpha = \beta = \delta = \frac{1}{2}\).
- \(hp\)-version, locally refined mesh: \(\alpha, \beta, \delta \) depend on local \(h, p\).
Variational formulation of the TDG

With this fluxes, summing over the elements $K \in \mathcal{T}_h$, the TDG method reads:

$$\text{find } u_p \in V_p(\mathcal{T}_h) \text{ s.t.}$$

$$A_h(u_p, v_p) = i\omega^{-1} \int_{\partial \Omega} \delta g \nabla_h v_p \cdot \mathbf{n} \, dS + \int_{\partial \Omega} (1 - \delta) g \overline{v_p} \, dS,$$

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\]
\[
\forall \, v_p \in V_p(\mathcal{T}_h), \text{ where } \quad (\mathcal{F}_h^I = \text{interior skeleton})
\]

\[
\mathcal{A}_h(u, v) := \int_{\mathcal{F}_h^I} \{u\} \overline{\nabla_h v} \cdot n \, dS + i\omega^{-1} \int_{\mathcal{F}_h^I} \beta \{\nabla_h u\} \overline{\nabla_h v} \cdot n \, dS
\]

\[
- \int_{\mathcal{F}_h^I} \{\nabla_h u\} \cdot \overline{\nabla_h v} \cdot n \, dS + i\omega \int_{\mathcal{F}_h^I} \alpha \{u\} \overline{\nabla_h v} \cdot n \, dS
\]

\[
+ \int_{\partial \Omega} (1 - \delta) u \nabla_h v \cdot n \, dS + i\omega^{-1} \int_{\partial \Omega} \delta \nabla_h u \cdot n \nabla_h v \cdot n \, dS
\]

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- \int_{\partial \Omega} \delta \nabla_h u \cdot n v \, dS + i\omega \int_{\partial \Omega} (1 - \delta) u v \, dS.
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With this fluxes, summing over the elements $K \in \mathcal{T}_h$, the TDG method reads:

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$\forall \, v_p \in V_p(\mathcal{T}_h),$ where

$$(\mathcal{F}^I_h = \text{interior skeleton})$$

$$A_h(u, v) := \int_{\mathcal{F}^I_h} \{ u \} \nabla_h v \cdot \mathbf{n} \, dS + i \omega^{-1} \int_{\mathcal{F}^I_h} \beta \nabla_h u \cdot \nabla_h v \cdot \mathbf{n} \, dS$$

$$- \int_{\mathcal{F}^I_h} \{ \nabla_h u \} \cdot [v] \, dS + i \omega \int_{\mathcal{F}^I_h} \alpha u \cdot [v] \, dS$$

$$+ \int_{\partial \Omega} (1 - \delta) u \nabla_h v \cdot \mathbf{n} \, dS + i \omega^{-1} \int_{\partial \Omega} \delta \nabla_h u \cdot \mathbf{n} \nabla_h v \cdot \mathbf{n} \, dS$$

$$- \int_{\partial \Omega} \delta \nabla_h u \cdot \mathbf{n} v \, dS + i \omega \int_{\partial \Omega} (1 - \delta) u \overline{v} \, dS.$$

$$u_p \mapsto (\text{Im} \, A_h(u_p, u_p))^{\frac{1}{2}} \text{ is a norm on the Trefftz space } \Rightarrow \exists! \, u_p.$$
Unconditional quasi-optimality

On the Trefftz space

\[ T(T_h) := \left\{ v \in L^2(\Omega) : v|_K \in H^2(K), -\Delta v - \omega^2 v = 0 \text{ in each } K \in T_h \right\}, \]

\[ \forall v, w \in T(T_h): \]
\[ \text{Im } A_h(v, v) = |||v|||_{\mathcal{F}_h}^2 \]

\[ |A_h(w, v)| \leq 2 |||w|||_{\mathcal{F}_h^+} |||v|||_{\mathcal{F}_h} \]

\[
\|\|u - u_p\|\|_{\mathcal{F}_h} \leq 3 \|\|u - v_p\|\|_{\mathcal{F}_h^+} \]

\[ \forall v_p \in V_p(T_h) \subset T(T_h). \]

Using norms

\[ \|\|v\|\|_{\mathcal{F}_h^+}^2 := \omega^{-1} \|\beta^{1/2} [\nabla_h v]_N\|_{0, \mathcal{F}_h^I}^2 + \omega \|\alpha^{1/2} [v]_N\|_{0, \mathcal{F}_h^I}^2 \]

\[ + \omega^{-1} \|\delta^{1/2} \nabla_h v \cdot n\|_{0, \partial \Omega}^2 + \omega \|(1 - \delta)^{1/2} v\|_{0, \partial \Omega}^2, \]

\[ \|\|v\|\|_{\mathcal{F}_h}^2 := \|\|v\|\|_{\mathcal{F}_h}^2 + \omega \|\beta^{-1/2} [v]_N\|_{0, \mathcal{F}_h^I}^2 \]

\[ + \omega^{-1} \|\alpha^{-1/2} [\nabla_h v]_N\|_{0, \mathcal{F}_h^I}^2 + \omega \|\delta^{-1/2} v\|_{0, \partial \Omega}^2. \]
Monk–Wang duality technique
\[ \| w \|_{L^2(\Omega)} \leq C(\omega, h, \Omega, T_h, \alpha, \beta, \delta) \| | w \|_{F_h} \forall w \in T(T_h) \]

→ quasi-optimality in \( L^2(\Omega) \)-norm.

Assume for now: best approximation estimates for plane or circular waves (shown later in this talk).
TDG $p$-convergence

Monk–Wang duality technique
\[ \| w \|_{L^2(\Omega)} \leq C(\omega, h, \Omega, T_h, \alpha, \beta, \delta) \| | w | |_{F_h} \quad \forall w \in T(T_h) \]
\[ \rightarrow \text{quasi-optimality in } L^2(\Omega)-\text{norm}. \]

Assume for now: best approximation estimates for plane or circular waves (shown later in this talk).

We obtain ($h$- and) $p$-estimates for plane/circular waves (2D):
\[ \| | u - u_p | |_{F_h} \leq C(\omega h) \omega^{-\frac{1}{2}} h^{k-\frac{1}{2}} \left( \frac{\log(p)}{p} \right)^{k-\frac{1}{2}} \| u \|_{k+1, \omega, \Omega}, \]
\[ \omega \| u - u_p \|_{L^2(\Omega)} \leq C(\omega h) \text{diam}(\Omega) h^{k-1} \left( \frac{\log(p)}{p} \right)^{k-\frac{1}{2}} \| u \|_{k+1, \omega, \Omega}, \]
on quasi-uniform meshes with meshsize $h$.

Slightly different orders of convergence in $p$ in 3D.
Numerical tests

Plane wave spaces, $\omega = 10$, $h = 1/\sqrt{2}$, $L^2$-norm of errors:

Smooth solution in $C^\infty(\mathbb{R}^2)$

$$u(x) = J_1(\omega|x|) \cos \theta$$

exponential convergence.

Singular solution in $H^{5/2-\epsilon}(\Omega)$

$$u(x) = J_{3/2}(\omega|x|) \cos(\frac{3}{2}\theta)$$

algebraic convergence.

Numerical instability / ill-conditioning for high $p$!
# The road map

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Part II

Approximation in Trefftz spaces
The best approximation estimates

The analysis of any plane wave Trefftz method requires best approximation estimates:

\[-\Delta u - \omega^2 u = 0 \quad \text{in } D \in \mathcal{T}_h, \quad u \in H^{k+1}(D),\]

\[\text{diam}(D) = h, \quad p \in \mathbb{N}, \quad d_1, \ldots, d_p \in \mathbb{S}^{N-1},\]

\[\inf_{\vec{\alpha} \in \mathcal{C}_p} \left\| u - \sum_{\ell=1}^{p} \alpha_\ell e^{i\omega d_\ell \cdot x} \right\|_{H^j(D)} \leq C \epsilon(h, p) \| u \|_{H^{k+1}(D)},\]

with explicit \( \epsilon(h, p) \xrightarrow{h \to 0} 0 \) and \( \epsilon(h, p) \xrightarrow{p \to \infty} 0 \).
The best approximation estimates

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\]

with explicit \( \varepsilon(h, p) \xrightarrow{h \to 0} 0 \) \( p \to \infty \).

Goal: precise estimates on \( \varepsilon(h, p) \)

- for plane and circular/spherical waves;
- both in \( h \) and \( p \) (simultaneously);
- in 2 and 3 dimensions;
- with explicit bounds in the wavenumber \( \omega \).
The Vekua theory in $N$ dimensions

We need an old (1940s) tool from PDE analysis: Vekua theory.

$D \subset \mathbb{R}^N$ star-shaped wrt. $0$, $\omega > 0$.

Define two continuous functions:

\[
M_1, M_2 : D \times [0, 1] \to \mathbb{R}
\]

\[
M_1(x, t) = -\frac{\omega |x|}{2} \frac{\sqrt{t}^{N-2}}{\sqrt{1-t}} J_1(\omega |x| \sqrt{1-t}),
\]

\[
M_2(x, t) = -\frac{i\omega |x|}{2} \frac{\sqrt{t}^{N-3}}{\sqrt{1-t}} J_1(i\omega |x| \sqrt{t(1-t)}).
\]

The Vekua operators

\[
V_1, V_2 : C^0(D) \to C^0(D),
\]

\[
V_j[\phi](x) := \phi(x) + \int_0^1 M_j(x, t)\phi(tx) \, dt, \quad \forall x \in D, j = 1, 2.
\]
4 properties of Vekua operators

1. \[ V_2 = (V_1)^{-1} \]

2. \[ \Delta \phi = 0 \iff (-\Delta - \omega^2) V_1[\phi] = 0 \]

Main idea of Vekua theory:

Harmonic functions \( \xrightarrow{V_2} \) Helmholtz solutions \( \xleftarrow{V_1} \)

3. Continuity in (\( \omega \)-weighted) Sobolev norms, explicit in \( \omega \)
   \[ (H^j(D), W^{j,\infty}(D), j \in \mathbb{N}) \]

4. \( P = \) Harmonic polynomial \iff \( V_1[P] = \) circular/spherical wave
   \[
   \begin{bmatrix}
   e^{it\psi} J_l(\omega r), \\
   Y_l^m(\frac{\mathbf{x}}{|\mathbf{x}|}) J_l(\omega |\mathbf{x}|)
   \end{bmatrix}
   \]
   \( 2D \)
   \( 3D \)
Vekua operators & approximation by GHPs

\[ -\Delta u - \omega^2 u = 0, \quad u \in H^{k+1}(D), \]

\[ \downarrow V_2 \]

\[ V_2[u] \text{ is harmonic} \implies \text{can be approximated by harmonic polynomials} \]

(harmonic Bramble–Hilbert in \( h \),
Complex analysis in \( p \)-2D (Melenk), new result in \( p \)-3D),

\[ \downarrow V_1 \]

\[ u \text{ can be approximated by GHPs:} \]

\[ \text{generalized harmonic polynomials} := V_1 \left[ \begin{array}{c} \text{harmonic polynomials} \end{array} \right] = \text{circular/spherical waves.} \]

(\( \rightarrow \) Bounds applicable to any GHP-based Trefftz schemes!)
The approximation of GHPs by plane waves

Link between plane waves and circular/spherical waves: 
Jacobi–Anger expansion

\[
2D \quad e^{iz \cos \theta} = \sum_{l \in \mathbb{Z}} i^l J_l(z) \ e^{il\theta} \quad z \in \mathbb{C}, \ \theta \in \mathbb{R},
\]

\[
3D \quad e^{ir \xi \cdot \eta} = 4\pi \sum_{l \geq 0} \sum_{m=-l}^{l} i^l j_l(r) \ Y_{l,m}(\xi) \overline{Y_{l,m}(\eta)} \quad \xi, \eta \in \mathbb{S}^2, \ r \geq 0.
\]

We need the other way round:

\[\text{GHP} \approx \text{linear combination of plane waves}\]

- truncation of J–A expansion,
- careful choice of directions (in 3D), \rightarrow explicit error bound,
- solution of a linear system,
- residual estimates,
The final approximation by plane waves

\[ -\Delta u - \omega^2 u = 0 \quad \xrightarrow{V_2} \quad -\Delta V_2[u] = 0 \]

harmonic approx. ↓

Circular waves ↓ (Jacobi–Anger)\(^{-1}\)

Plane waves

Final estimate

\[
\inf_{\alpha \in \mathbb{C}^p} \left| u - \sum_{\ell=1}^{p} \alpha_\ell e^{i\omega x \cdot d_\ell} \right|_{j,\omega,D} \leq C(\omega h) h^{k+1-j} q^{-\lambda(k+1-j)} \| u \|_{k+1,\omega,D}
\]

In 2D: \( p = 2q + 1 \), \( \lambda(D) \) explicit, \( \forall d_\ell \).

In 3D: \( p = (q+1)^2 \), \( \lambda(D) \) unknown, special \( d_\ell \).

better than poly.!

If \( u \) extends outside \( D \): exponential order in \( q \). (Same for GHPs.)
The road map

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Part III

What about $hp$-TDG?
What do we want?

*hp*-convergence is achieved by combination of mesh refinement and increase of #DOFs per element.

Typical *hp*-result on a priori graded meshes for Laplace 2D:

\[ \| u - u_{hp} \|_{H^1(\Omega)} \leq Ce^{-b \sqrt[3]{\#DOFs}} \quad C, b > 0.\]
What do we want?

\(hp\)-convergence is achieved by combination of mesh refinement and increase of #DOFs per element.

Typical \(hp\)-result on a priori graded meshes for Laplace 2D:

\[
\| u - u_{hp} \|_{H^1(\Omega)} \leq C e^{-b \sqrt[3]{\text{#DOFs}}} \quad C, b > 0.
\]

We prove, for TDG + plane/circular wave basis, Helmholtz 2D:

\[
\| u - u_{hp} \|_{L^2(\Omega)} \leq C e^{-b \sqrt{\text{#DOFs}}} \quad C, b > 0.
\]
What do we need?

Consider 2D Helmholtz impedance (+Dirichlet) BVP, with piecewise analytic domain $\Omega$ and boundary conditions $g$.

So far we have proved:

◮ unconditional well-posedness and quasi-optimality,
◮ approximation bounds in $h$ and $p$ simultaneously.

What else do we need to obtain exponential convergence?

◮ specify meshes and fluxes (modify duality);
◮ analytic regularity and extendibility of solutions;
◮ improved approximation bounds...
Explicit dependence on element geometry

Polynomial FEM: best approximation bounds on $K \in \mathcal{T}_h$ obtained by scaling to reference element $\hat{K}$.

\[ \Delta u + \omega^2 u = 0 \text{ in } K, \quad \rightarrow \quad \text{pullback } \hat{u}(\hat{x}) := u(F(\hat{x})) \text{ is not Trefftz} \]
\[ \rightarrow \quad \text{not approximable by Trefftz basis.} \]

Even for affine scaling:
\[ P^q(\hat{K}) \rightarrow P^q(K) \]
\[ PW^q(\hat{K}) \rightarrow ?? \]

Every element $K$ has “its own” approximation bound
\[ \rightarrow \quad \text{constants depend on the shape of } K \quad \rightarrow \quad \text{(in principle) not uniformly bounded on unstructured graded meshes}. \]

We want “universal bounds” independent of the geometry, but...
Explicit dependence on element geometry

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$$\Delta u + \omega^2 u = 0 \text{ in } K \quad \Rightarrow \quad \text{pullback } \hat{u}(\hat{x}) := u(F(\hat{x})) \text{ is not Trefftz}$$

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Every element $K$ has “its own” approximation bound

$$\Rightarrow \quad \text{constants depend on the shape of } K \quad \Rightarrow \quad \text{(in principle) not uniformly bounded on unstructured graded meshes.}$$

We want “universal bounds” independent of the geometry, but... we get more: **fully explicit bounds** for curvilinear non-convex elements.
Assumption and tools

Assumption on element $D$: (Very weak!)

- $D \subset \mathbb{R}^2$ s.t. $\text{diam}(D) = 1$, star-shaped wrt $B_\rho$, $0 < \rho < 1/2$.

Define:

- $D_\delta := \{z \in \mathbb{R}^2, d(z, D) < \delta\}$,
- $\xi := \left\{ \begin{array}{ll}
1 & D \text{ convex}, \\
\frac{2}{\pi} \arcsin \frac{\rho}{1-\rho} & < 1.
\end{array} \right.$

Use:

- M. Melenk’s ideas;
- complex variable, identification $\mathbb{R}^2 \leftrightarrow \mathbb{C}$, harmonic $\leftrightarrow$ holomorphic;
- conformal map level sets, Schwarz–Christoffel;
- Hermite interpolant $q_n$;
- lot of “basic” geometry and trigonometry, nested polygons, plenty of pictures...
Explicit approximation estimate

**Approximation result**

Let \( n \in \mathbb{N}, \) \( f \) holomorphic in \( D_\delta, \) \( 0 < \delta \leq 1/2, \)

\[ h := \min \left\{ \left( \frac{\xi \delta}{27} \right)^{1/\xi} / 3, \frac{\rho}{4} \right\}, \quad \Rightarrow \quad \exists q_n \text{ of degree } \leq n \text{ s.t.} \]

\[ \| f - q_n \|_{L^\infty(D)} \leq 7 \rho^{-2} \ h^{-\frac{72}{\rho^4}} (1 + h)^{-n} \| f \|_{L^\infty(D_\delta)}. \]

- Fully explicit bound;
- **exponential** in degree \( n; \)
- \( h \geq \text{"conformal distance"}(D, \partial D_\delta), \) related to physical dist. \( \delta; \)
- in convex case \( h = \min\{\delta/27, \rho/4\}; \)
- extends to harmonic \( f/q_n \) and derivatives \((W^{j,\infty}-\text{norm});\)
- extended to Helmholtz solutions and circular/plane waves (fully explicit \( W^{j,\infty}(D)-\text{continuity of Vekua operators}).\)
Sequence of meshes with:

- element diameters $h_K$ geometrically graded (with $0 < \sigma < 1$) towards domain corners;
- any star-shaped element allowed! $K$ star-shaped wrt $B_{\rho h_K}(x_K)$.

$\rho$ and $\sigma$ are important parameters in the convergence.

Increase #DOFs by simultaneously:

1. refining layer of small elements,
2. increasing number of PWs/CWs in each element.
We simply choose the flux parameters \( (h_K := \text{diam } K) \)

\[
\alpha = a \frac{\max_{K \in \mathcal{T}_h} h_K}{\min\{h_{K_1}, h_{K_2}\}} \quad \text{on } K_1 \cap K_2, \quad a, \beta, \delta > 0 \text{ constant.}
\]

This choice gives “balance” between approximation and duality.

To guarantee shape-independence, we develop new trace estimates with explicit dependence on the element geometry through the parameter \( \rho \).
Approximation in the elements

Need to bound \( \inf_{v_p \in V_p} \| u - v_p \| \) in two cases:

1. Exponentially small elements at domain corners.
   Use that in tiny elements PWs / CWs behave like \( P_1 \) polynomials.
   Difficulty: \( \nabla u \not\in L^\infty, \ u \not\in H^2 \).

\[ h_K \sim d(K, \text{corners}) \sim d(K, \partial(\text{analyticity region of } u)) \forall K. \]

\[ \Rightarrow \text{we can use previous explicit bounds.} \]

Putting everything together: desired exponential convergence
\[ \| u - u_{hp} \| \leq C e^{-b \sqrt{\# \text{DOFs}}} \]
\( C, b > 0 \).
Approximation in the elements

Need to bound \( \inf_{v_p \in V_p} \| u - v_p \| \) in two cases:

1. **Exponentially small elements at domain corners.**
   Use that in tiny elements PWs / CWs behave like \( \mathbb{P}^1 \) polynomials.
   Difficulty: \( \nabla u \notin L^\infty, \ u \notin H^2 \).

2. **Larger elements away from corners.**
   Following Melenk, \( u \in B^{2, \frac{1}{1+\omega}}_\beta (\Omega) \), weighted countably-normed space, and extends analytically (similar to Laplace solutions):
   \[ h_K \sim d(K, \text{corners}) \sim d(K, \partial(\text{analyticity region of } u)) \quad \forall K. \]
   \[ \Rightarrow \ 	ext{we can use previous explicit bounds.} \]
Approximation in the elements

Need to bound $\inf_{v_p \in V_p} \| u - v_p \|$ in two cases:

1. **Exponentially small elements at domain corners.**
   Use that in tiny elements PWs / CWs behave like $\mathbb{P}^1$ polynomials.
   Difficulty: $\nabla u \notin L^\infty, \ u \notin H^2$.

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   Following Melenk, $u \in B^2_{\beta,1+\omega}(\Omega)$, weighted countably-normed space, and extends analytically (similar to Laplace solutions):
   \[
   \Rightarrow \quad h_K \sim d(K, \text{corners}) \sim d(K, \partial(\text{analyticity region of } u)) \quad \forall K.
   \]
   \[
   \Rightarrow \quad \text{we can use previous explicit bounds.}
   \]

Putting everything together: desired exponential convergence

\[
\| u - u_{hp} \|_{L^2(\Omega)} \leq Ce^{-b \sqrt{\# \text{DOFs}}} \quad C, b > 0.
\]
<table>
<thead>
<tr>
<th></th>
<th>Helmholtz</th>
<th>Maxwell</th>
</tr>
</thead>
<tbody>
<tr>
<td>Formulation of TDG</td>
<td>✓</td>
<td>~ Helm.</td>
</tr>
<tr>
<td>TDG $|||_{\mathcal{F}_h}$-quasi optimality</td>
<td>✓</td>
<td>~ Helm.</td>
</tr>
<tr>
<td>Duality argument</td>
<td>$L^2(\Omega)$</td>
<td>$H(\text{div}, \Omega)'$</td>
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<tr>
<td>$hp$ exponential convergence</td>
<td>✓ (2D)</td>
<td>×</td>
</tr>
<tr>
<td>Approximation by GHPs</td>
<td>✓</td>
<td>✓ ($p$ non sharp)</td>
</tr>
<tr>
<td>Approximation by PWs</td>
<td>✓</td>
<td>✓ (non sharp)</td>
</tr>
</tbody>
</table>
Summary and open problems

What we have done:

◮ **TDG formulation**, unconditional well-posedness;
◮ **approximation** theory: holomorphic, harmonic, Helmholtz;
◮ **$h$- and $p$-convergence** for plane and spherical waves;
◮ exponential **$hp$-convergence** on graded meshes in 2D;
◮ (not discussed: extensions to **Maxwell equations**).

Plenty of possible research directions:

non-constant coefficients $ω(\mathbf{x})$;

adaptivity in PW directions;

other PDEs, time-domain;

new bases;

defeat ill-conditioning, . . .

Thank you!
Our bibliography

(Helmholtz)


(Maxwell)


(Elasticity)
